

Gravitational wave measurements and the breaking of parallelograms in space-time

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Abstract

We consider a simple experiment to detect plane-fronted gravitational waves, similar to the experiments that are expected to lead to gravitational wave measurements. In the present setup, an incident beam of light splits into two beams running along perpendicular arms, in similarity to the arms of the standard gravitational wave interferometers. The two arms have the same length, and each one has a fixed mirror at the end. The reflected light beams are detected at the same point of the splitting. Along each arm of the setup, the two light beams define two null vectors in space-time: the forward vector and the reflected vector. We show that the sum of these four vectors, the forward and reflected null vectors along the two arms, do form a parallelogram in flat space-time, but not in the presence of a plane-fronted gravitational wave. The non-closure of the parallelogram expresses the existence of gravitational waves, and is a manifestation of the torsion of the space-time. The present setup is an alternative and well suited procedure that may improve the detection of gravitational waves.

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1 Introduction

Gravitational waves are one of the most important consequences of general relativity. They are due to the dynamical nature of space-time, which is described by Einstein's equations. It is natural to consider a gravitational wave as a kind of ripple in the background flat geometry. Gravitational waves may be classified either as non-linear waves, which are exact solutions of Einstein's equations, or linearized waves, which are solutions of the linearized Einstein's equations, presented in standard textbooks on general relativity. One of the simplest realizations of a non-linear gravitational wave is given by the solution known as plane-fronted gravitational wave, studied by Ehlers and Kundt [1].

The present attempts to observe gravitational waves are based on laser interferometers, constituted by two long and perpendicular arms that are able, in principle, to detect gravitational waves travelling in the direction normal the plane formed by the two arms (see, for instance, Refs. [2, 3]). The mirrors at the end of the long arms are attached to test masses that are hung from wires, and are free to swing in the horizontal directions. In contrast, in our setup the mirrors are *fixed* at the end of the arms.

In this article, we will consider laser beams that travel back and forth along the arms, in the presence of a plane-fronted gravitational wave travelling in the z direction. Let us denote by v^μ and w^μ the null vectors that represent, respectively, (i) the laser beam trajectory from the splitting point S to the fixed mirror M_1 , and (ii) from M_1 back to S , along the x direction. Along the y direction, the vectors a^μ and b^μ travel (iii) from the splitting point S to the fixed mirror M_2 , and (iv) from M_2 back to S , respectively. In flat space-time, these four null vectors form a parallelogram in the sense that $v^\mu + w^\mu = a^\mu + b^\mu$. We will show in this analysis that in the presence of a plane-fronted gravitational wave, these null vectors no longer form a parallelogram, because $v^\mu + w^\mu \neq a^\mu + b^\mu$. This fact is an indication that the space-time has torsion. By establishing the tetrad frame adapted to stationary observers in the space-time of plane-fronted gravitational waves, we will obtain the torsion tensor that precisely explains the breaking of the parallelogram. The result described here is geometrically similar to the breaking of parallelograms in the Schwarzschild space-time, in the context of the Pound-Rebka experiment [4, 5]. The breaking of parallelograms shows that torsion is an intrinsic geometric property of space-time, and indicates the presence of gravitational waves.

The existence of gravitational waves is verified by the lapse of time between the arrival of the two reflected (perpendicular) light beams. As we will show, it may be experimentally easier to measure this time interval, since the mirrors are fixed at the end of the arms, than to detect electromagnetic interference in the measurement of linearized gravitational waves by means of the current approaches.

Notation: space-time indices μ, ν, \dots and $\text{SO}(3,1)$ indices a, b, \dots run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted $e^a{}_\mu$, and the torsion tensor reads $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, Minkowski space-time metric tensor is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. We assume that the space-time geometry is defined by the tetrad field only, and in this case the only possible non-trivial definition of the torsion tensor is given by $T^a{}_{\mu\nu}$.

2 The breaking of parallelograms in space-time

We will consider the space-time of a plane-fronted gravitational wave. The space-time is described by a line element that is an exact solution of Einstein's equations, and is given by [1, 6]

$$ds^2 = dx^2 + dy^2 + 2du\,dv + H(x, y, u)du^2. \quad (1)$$

The function $H(x, y, u)$ satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) H(x, y, u) = 0. \quad (2)$$

Transforming the (u, v) to (t, z) coordinates, where

$$u = \frac{1}{\sqrt{2}}(z - t), \quad v = \frac{1}{\sqrt{2}}(z + t),$$

we find

$$ds^2 = \left(\frac{H}{2} - 1 \right) dt^2 + dx^2 + dy^2 + \left(\frac{H}{2} + 1 \right) dz^2 - H\,dt\,dz. \quad (3)$$

The function H is required to satisfy only Eq. (2). However, it would be interesting to specify H such that it describes a wave-packet [6].

Before we investigate the action of the plane-fronted gravitational waves in the experimental devices, let us establish the experimental setup first in flat space-time. We will consider a laser beam in the xy plane that splits into two beams: one beam along an arm of proper length L , in the x direction, and another beam along a similar arm, also with length L , in the y direction. The splitting point is denoted by S , and has coordinates (x_0, y_0) . At the ends of each arm there are fixed mirrors, M_1 and M_2 , that reflect the light beams back to the splitting point S . The experimental setup is displayed in Figure 1.

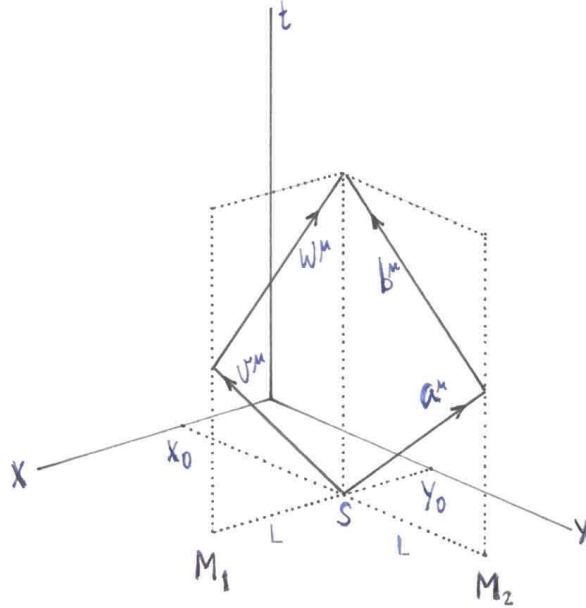


Figure 1: The parallelogram

The trajectory of a light ray in the x direction is given by a null vector $v^\mu(t, x, y, z) = (v^0, v^1, 0, 0)$ that satisfies the condition $v^\mu v^\nu g_{\mu\nu} = 0$. For a light ray travelling in the positive direction along the x axis, the latter

condition yields $v^1 = (-g_{00}/g_{11})^{1/2} v_0 = (1 - H/2)^{1/2} v_0$. We will fix the value of v^1 by means of the following procedure. For a vector v^μ that represents a light ray between the point S and the position M_1 of the mirror, we require that v^1 has length L , in the presence of a plane fronted gravitational wave. Thus we require

$$v^1 = \int_{x_0}^{x_0+L} dx \sqrt{g_{11}} = L. \quad (4)$$

It follows that

$$v^0 = \frac{L}{(1 - H/2)^{1/2}}, \quad H = H(x^0 + L, y_0, t_0 + dt), \quad (5)$$

where t_0 is the time coordinate of the emitted light ray at the point S , and $t_0 + dt$ is the time coordinate when the light ray reaches M_1 . In flat space-time, the reflected light ray returns to the point S in the time coordinate $t_0 + 2dt$. Therefore the vector v^μ is defined by

$$v^\mu = \left(\frac{L}{(1 - H/2)^{1/2}}, L, 0, 0 \right), \quad H = H(x_0 + L, y_0, t_0 + dt). \quad (6)$$

We define the reflected vector w^μ , that travels from M_1 back to S , according to

$$w^\mu = \left(\frac{L}{(1 - H/2)^{1/2}}, -L, 0, 0 \right), \quad H = H(x_0, y_0, t_0 + 2dt). \quad (7)$$

Along the y direction, the vectors a^μ and b^μ that travel from S to the mirror M_2 , and from M_2 back to S , are constructed in similarity to v^μ and w^μ . They read

$$a^\mu = \left(\frac{L}{(1 - H/2)^{1/2}}, 0, L, 0 \right), \quad H = H(x_0, y_0 + L, t_0 + dt), \quad (8)$$

$$b^\mu = \left(\frac{L}{(1 - H/2)^{1/2}}, 0, -L, 0 \right), \quad H = H(x_0, y_0, t_0 + 2dt). \quad (9)$$

The vectors v^μ , w^μ , a^μ and b^μ are defined for a fixed coordinate position z_0 . They are located at $(x_0 + L, y_0, t_0 + dt)$, $(x_0, y_0, t_0 + 2dt)$, $(x_0, y_0 + L, t_0 + dt)$

and $(x_0, y_0, t_0 + 2dt)$, respectively. However, we find it suitable to depict them as in Fig. 1, since they do form a parallelogram in a flat geometry.

Let us define the breaking of the parallelogram in space-time by the vector

$$\Delta^\mu = (v^\mu + w^\mu) - (a^\mu + b^\mu). \quad (10)$$

It is easy to see that for $i = 1, 2, 3$ we have $\Delta^i = 0$, and that for Δ^0 we have

$$\begin{aligned} \Delta^0 = L \Big\{ & \frac{1}{[1 - \frac{1}{2}H(x_0 + L, y_0, t_0 + dt)]^{1/2}} \\ & + \frac{1}{[1 - \frac{1}{2}H(x_0, y_0, t_0 + 2dt)]^{1/2}} \\ & - \frac{1}{[1 - \frac{1}{2}H(x_0, y_0 + L, t_0 + dt)]^{1/2}} \\ & - \frac{1}{[1 - \frac{1}{2}H(x_0, y_0, t_0 + 2dt)]^{1/2}} \Big\} \end{aligned} \quad (11)$$

The second and fourth terms above cancel each other, and we are left with

$$\Delta^0 = L \frac{\{[1 - \frac{1}{2}H(x_0, y_0 + L, t_0 + dt)]^{1/2} - [1 - \frac{1}{2}H(x_0 + L, y_0, t_0 + dt)]^{1/2}\}}{[1 - \frac{1}{2}H(x_0 + L, y_0, t_0 + dt)]^{1/2}[1 - \frac{1}{2}H(x_0, y_0 + L, t_0 + dt)]^{1/2}}. \quad (12)$$

The only assumption we make in this analysis is that L is sufficiently small, so that we can expand H in terms of L according to

$$H(x_0 + L, y_0, t_0) \approx H(x_0, y_0, t_0) + \frac{\partial H(x_0, y_0, t_0)}{\partial x} L \equiv H + \frac{\partial H}{\partial x} L. \quad (13)$$

Likewise,

$$H(x_0, y_0 + L, t_0) \approx H(x_0, y_0, t_0) + \frac{\partial H(x_0, y_0, t_0)}{\partial y} L \equiv H + \frac{\partial H}{\partial y} L. \quad (14)$$

We may consider the function H to represent a wave packet in the form of a Gaussian function, that travels in the z direction. The approximation above holds if L is much smaller compared to the width of the Gaussian (or

if H varies very weakly over the space region of Fig. 1). In view of the approximation above, Eq. (12) is finally simplified to

$$\Delta^0 = \frac{1}{4}L^2 \left[\frac{\left(\frac{\partial H}{\partial x}\right) - \left(\frac{\partial H}{\partial y}\right)}{\left(1 - \frac{1}{2}H\right)^{3/2}} \right]. \quad (15)$$

The equation above expresses the non-closure in space-time of the parallelogram displayed in Figure 1, and thus indicates the presence of torsion in the space-time geometry.

3 The frame and the torsion tensor

We will present here the torsion tensor that precisely explains the breaking of the parallelogram of Fig. 1. For this purpose, we need to establish the reference frame for the measurement of Eq. (15). This reference frame is the same one adopted in Ref. [7], in the analysis of the energy-momentum of plane-fronted gravitational waves. Let us denote by $x^\mu(s)$ the worldline C of an arbitrary observer in space-time, and by $u^\mu = dx^\mu/ds$ its velocity along C . We identify the timelike component $e_{(0)}^\mu$ of the the inverse tetrad field with the velocity of the observer, i.e., $e_{(0)}^\mu = u^\mu$. The components $e_{(i)}^\mu$, for $i = 1, 2, 3$ may be specified by requiring these spacelike vectors to be oriented asymptotically, for instance, along the three Cartesian axes x, y, z . Reference frames may be alternatively characterized by means of the acceleration tensor. These are the inertial accelerations (translational and rotational) that are necessary to maintain the frame in a given inertial state in space-time. This issue has been discussed in Refs. [5, 8]. In the present investigation, the tetrad frame established in Ref. [7] is well suited for our purposes.

An stationary observer in space-time is characterized by the condition $e_{(0)}^i = 0$. A suitable construction of a tetrad frame, adapted to the stationary character of the observer, is [7]

$$e_{a\mu} = \begin{pmatrix} -A & 0 & 0 & -B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & C \end{pmatrix}, \quad (16)$$

where

$$A = \left(1 - \frac{H}{2}\right)^{1/2}, \quad AB = \frac{H}{2}, \quad AC = 1. \quad (17)$$

In (16), a and μ label rows and columns, respectively. It is not difficult to verify that Eq. (16) yields [7]

$$e_{(0)}{}^\mu = (1/A, 0, 0, 0), \quad (18)$$

and

$$\begin{aligned} e_{(1)}{}^\mu &= (0, 1, 0, 0), \\ e_{(2)}{}^\mu &= (0, 0, 1, 0), \\ e_{(3)}{}^\mu &= (-H/(2A), 0, 0, A). \end{aligned} \quad (19)$$

Note that if $H \ll 1$ we have $A \cong 1 - H/4$ and therefore $e_{(3)}{}^i = (0, 0, A) \cong (0, 0, 1 - H/4)$.

The frame is determined by fixing six conditions on $e_{a\mu}$. Equation (18) fixes the kinematic state of the frame, since the three velocity conditions $e_{(0)}{}^i = 0$ ensure that the frame is stationary. Three other conditions fix the spatial orientation of the frame. According to Eq. (19), $e_{(1)}{}^\mu$, $e_{(2)}{}^\mu$ and $e_{(3)}{}^\mu$ are unit vectors along the x , y and z axis, respectively. Note that by requiring $H = 0$ we obtain $e_a{}^\mu = \delta_a^\mu$, and consequently $T_{a\mu\nu} = 0$. The nonvanishing components of $T_{\mu\nu\lambda}$ are given in Eq. (19) of Ref. [7].

The breaking of parallelograms in a space-time with torsion is analysed by considering two vectors, $A^\mu = dx^\mu$ and $B^\mu = \delta x^\mu$. The parallel transport of A^μ along δx^μ , and of B^μ along dx^μ are given by, respectively,

$$\delta A^\mu = -\Gamma_{\alpha\beta}^\mu A^\alpha \delta x^\beta, \quad \delta B^\mu = -\Gamma_{\alpha\beta}^\mu B^\alpha dx^\beta, \quad (20)$$

where $\Gamma_{\alpha\beta}^\mu$ is an arbitrary space-time connection, with no *a priori* symmetry. The vectors $[A^\mu + (B^\mu + \delta B^\mu)]$ and $[B^\mu + (A^\mu + \delta A^\mu)]$ do not form a closed parallelogram if the space-time is endowed with torsion (see Fig.2). The breaking of a parallelogram was considered in Ref. [5], in the analysis of the Pound-Rebka experiment in the Schwarzschild space-time. In Fig. 2, the non-closure of the parallelogram is described by

$$\begin{aligned} [A^\mu + (B^\mu + \delta B^\mu)] - [B^\mu + (A^\mu + \delta A^\mu)] &= (\Gamma_{\alpha\beta}^\mu - \Gamma_{\beta\alpha}^\mu) dx^\alpha \delta x^\beta \\ &= T^\mu{}_{\alpha\beta} dx^\alpha \delta x^\beta. \end{aligned} \quad (21)$$

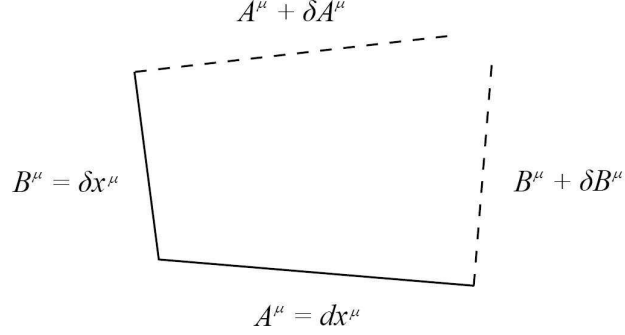


Figure 2: The breaking of the parallelogram

In the context of the previous Section we considered the arm length L to be small compared to the width of the wave packet that describes the function H . This property was taken into account in Eqs. (13) and (14). By comparing Figs. 1 and 2 we are led to identify $a^\mu = dx^\mu$ and $v^\mu = \delta x^\mu$. Therefore,

$$T^0_{\alpha\beta} dx^\alpha \delta x^\beta = T^0_{\alpha\beta} a^\alpha v^\beta. \quad (22)$$

A straightforward calculation leads to

$$T^0_{\alpha\beta} a^\alpha v^\beta = T^0_{01} a^0 v^1 - T^0_{02} a^2 v^0 - T^0_{12} a^2 v^1. \quad (23)$$

Taking into account the expressions of the torsion tensor components given in Ref. [7], and using $g^{00} = (-1 - \frac{1}{2}H)$ and $g^{03} = -\frac{1}{2}H$, we obtain

$$T^0_{\alpha\beta} a^\alpha v^\beta = \frac{1}{4} L^2 \left[\frac{(\frac{\partial H}{\partial x}) - (\frac{\partial H}{\partial y})}{(1 - \frac{1}{2}H)^{3/2}} \right], \quad (24)$$

and also $T^i_{\alpha\beta} a^\alpha v^\beta = 0$, for $i = 1, 2, 3$. Equation (24) represents precisely the breaking of the parallelogram given by Eq. (15).

4 Conclusions

The breaking of the parallelogram depicted in Fig. 1 was obtained by considering first the physical vectors v^μ , w^μ , a^μ and b^μ , as discussed in Section 2, and then by analysing the parallel transport of vectors in Section 3, in

a space-time endowed with torsion. The tetrad frame (16) and the torsion tensor components explain the breaking of the parallelogram. Therefore the measurement of the breaking of the parallelogram of Fig. 1 is, at the same time, a direct manifestation of gravitational waves, and of the space-time torsion.

The quantity Δ^0 given by Eq. (15) has dimension of length, and is measured on the worldline of the splitting point S . The interval of time elapsed between the arrival at the point S of the two light rays, described by w^μ and b^μ , is given by $\Delta\tau = \Delta^0/c$,

$$\Delta\tau = \frac{1}{4c} L^2 \left[\frac{(\frac{\partial H}{\partial x}) - (\frac{\partial H}{\partial y})}{(1 - \frac{1}{2}H)^{3/2}} \right], \quad (25)$$

where c is the speed of light.

The length L of one arm of the LIGO detector is 4 km . Assuming that the light rays perform 200 round trips along each arm (after the splitting at the point S and before the detection), and also assuming the amplitude of the gravitational wave to be small, we find

$$\Delta\tau \approx (533\text{ m} \cdot \text{s}) \left(\frac{\partial H}{\partial x} - \frac{\partial H}{\partial y} \right). \quad (26)$$

For a linearized gravitational wave it is assumed that the amplitude h of the wave is of the order $h \sim 10^{-21}$. The measurement of $\Delta\tau$ given by Eq. (26) may be carried out with an improved precision, compared to the required sensitivities for the detection of linearized waves, provided both $\partial H/\partial x$ and $\partial H/\partial y$ are of order less than 10^{-21}m^{-1} . We recall that in the present setup the mirrors are fixed at the end of the arms. Note that since H is a function of x, y and $u = (z - t)/\sqrt{2}$, the partial derivatives are not linearly dependent on the frequency of the incident gravitational wave. The measurement of $\Delta\tau$ may provide clues to the determination of H . In any case, the procedure presented here represents an alternative and promising way of measuring gravitational waves.

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